Robust quantum gates and a bus architecture for quantum computing with rare-earth-ion-doped crystals

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We present a composite pulse controlled phase gate which, together with a bus architecture, improves the feasibility of a recent quantum computing proposal based on rare-earth-ion-doped crystals. The proposed gate operation is tolerant to variations between ions of coupling strengths, pulse lengths, and frequency shifts. In the absence of decoherence effects, it achieves worst case fidelities above 0.999 with relative variations in coupling strength as high as 10% and frequency shifts up to several percent of the resonant Rabi frequency of the laser used to implement the gate. We outline an experiment to demonstrate the creation and detection of maximally entangled states in the system.

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I. INTRODUCTION

Several proposals have been made for quantum computing systems based on rare-earth ions embedded in cryogenic crystals and addressed by spectral hole burning techniques. This interest is motivated by several characteristics of crystal-embedded rare-earth ions that are highly desirable from the point of view of quantum information processing [1,2]. First, the hyperfine sublevels of the ion ground state serve as excellent quantum registers with a lifetime of hours and decoherence times up to several milliseconds. Second, the ions have large static dipole moments with interaction energies up to several gigahertz allowing controlled qubit interaction. In addition, optical transitions from the ground state, with homogeneous line widths of the order of kilohertz, are inhomogeneously broadened to several gigahertz, allowing us to address a large number of independent frequency channels.

In this paper we focus on the rare-earth quantum computer (REQC) proposal by Kröll and co-workers [3,4]. A key feature of this proposal is the use of an initialization process to select a macroscopic number of identical *instances* of a chosen quantum computer within the crystal. Each instance has exactly one representative from every active frequency *channel* within the inhomogeneous profile of the optical transition used to manipulate the ions. Furthermore, the dipole interaction strengths between the members of an instance are required to be above a chosen threshold value.

Due to finite channel width, ions representing the same channel in different instances will have slightly different inhomogeneous shifts. Similarly, the ion-field and ion-ion coupling strengths will differ between instances. Since ensemble quantum computing requires nearly identical evolution of all participating instances, we need to employ gate operations that are insensitive to such differences. In this paper, we demonstrate how such gate operations can be implemented by means of composite pulses and phase compensating operations.

This paper is structured as follows. In Sec. II, we briefly outline aspects of the REQC proposal of relevance to the present paper. Section III presents our proposal for high-fidelity gate operations based on composite pulses. Section IV discusses the possible benefits of replacing the cluster coupling topology assumed in the original REQC proposal with a bus based topology. Finally, Sec. V suggests a demonstration experiment illustrating the central ideas of REQC.

II. QUANTUM COMPUTING WITH RARE-EARTH IONS

In this section, we describe two central ideas of REQC, as described in Ref. [3]: the dynamical architecture selection and the controlled phase gate.

A. Dynamical architecture selection

The architecture of the REQC system is selected at start up by an initialization procedure. The desired end point of this process is a large number of independent instances of the chosen quantum computer, each instance being a group of ions with one representative from each active channel and couplings between the ions as required by the chosen architecture.

The initialization proceeds in two steps: channel preparation and identification of quantum computer instances. In both these steps, unwanted ions are deactivated by transferring them to off resonant, metastable states.

1. Channel preparation

A channel refers to a large number of ions distributed throughout the crystal, all having the same inhomogeneous shift and coupling strength within the inhomogeneously broadened optical transition used to access the ions. The channel preparation aims to deactivate all dopant ions close to resonance with a given channel and to transfer all members of the channel itself to their $|0\rangle$ state.

This can be achieved by means of spectral hole burning techniques, and widths of the final channel structure as low as 50 kHz have been obtained experimentally for materials similar to those considered for use in REQC [2].

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2. Instance identification

After a successful initialization, each ion will only be interacting with ions from other channels, allowing us to ignore ''excitation hopping'' transitions [1], as these will not be energy conserving. As a consequence, we can model the dipole coupling as simple couplings between the excited states:

$$V_{\text{dipole}}=\frac{1}{2}\sum_{\mu\neq\nu} g_{\mu\nu}(|e\rangle\langle e|)_{\mu}\otimes(|e\rangle\langle e|)_{\nu}, \tag{1}$$

where the sum is over all pairs of ions. To be precise about the objectives of the instance identification process, we will consider ions μ and ν to be coupled if $g_{\mu\nu}$ exceeds a threshold g_t determined by the chosen implementation of the gate operation.

The goal of the instance identification procedure is to transfer ions, which are in an active channel but are not members of a valid instance, to their auxiliary state $|\text{aux}\rangle$. One way to achieve this is to go through the following procedure for each pair (i,j) of channels required to be coupled.

By applying a π pulse to ions in channel i, we transfer the $|0\rangle$ population to the $|e\rangle$ state, thus shifting the excited state energy of all ions coupled to a channel i ion. By means of a frequency sweep or a comb of π rotations, all channel j ions which are shifted less than g_t are now transferred to their excited state $|e\rangle$, after which the channel i ions are returned to $|0\rangle$. We now wait for the excited channel j ions to decay, which will transfer a part of the ions to the inactive $|\text{aux}\rangle$ state.

Through repeated application of the pulse sequence described above, we can deactivate an arbitrarily high fraction of the channel j ions which are not coupled to a channel i ion. After this has been achieved, we repeat the process with the roles of channels i and j interchanged, and afterward proceed to apply the same procedure to all other edges of the coupling graph to finally arrive at the desired initialized REQC system.

B. Gate operation

In general, the coupling strengths $g_{\mu\nu}$ will differ between instances, requiring us to use gate operations that do not depend on the precise magnitude of the coupling strength.

One gate operation with this quality is the controlled phase shift based on the dipole blockade effect [5]. Assuming all ions not participating in the operation to be in their qubit states $|0\rangle$ and $|1\rangle$ and thus decoupled from the operation, we can implement a controlled phase shift in its simplest form by the following pulse sequence:

$$P_{0e}^{(i)}(\pi,\pi)P_{1e}^{(j)}(0,2\pi)P_{0e}^{(i)}(0,\pi), \tag{2}$$

with $P_{ab}^{(i)}(\phi,\theta)$ representing the effect of a resonant pulse of area θ and phase ϕ applied on the $|a\rangle$-$|b\rangle$ transition of ions in channel i.

For two coupled ions μ and ν, residing in channels i and j, respectively, the effect of performing the pulse sequence (2) would be the following. If ion μ is initially in the $|0\rangle_{\mu}$ state, the $P_{0e}^{(i)}(0,\pi)$ pulse transfers the ion to the excited state $|e\rangle_{\mu}$ and thus shifts the $P_{1e}^{(j)}(0,2\pi)$ pulse out of the resonance, causing the system to return to the initial state after the last π pulse. If, on the other hand, ion μ is initially in the $|1\rangle_{\mu}$ state, it is not transferred to the excited state $|e\rangle_{\mu}$ and thus the 2π pulse is resonant and causes a π phase shift on the $|1\rangle_{\nu}$ state. The effect of the full gate operation on the qubit space is consequently a π phase shift on the $|11\rangle$ state: $U_{\text{CPS}}=1-2|11\rangle\langle 11|$.

III. HIGH-FIDELITY GATE OPERATIONS

Gate operations for the REQC system face a number of challenges due to the fact that they operate simultaneously on a number of not quite identical instances of a quantum computer. Due to the finite channel width, ion μ will, in general, be detuned by a small amount $\delta^{(\mu)}$ from the central channel frequency. Furthermore, the experienced Rabi frequency $\Omega_0^{(\mu)}$ will differ slightly from the average Rabi frequency Ω_0 due to laser field inhomogeneities and local variations in dipole moments.

In this section, we will show that by taking advantage of the fact that $\delta^{(\mu)}$ and $\Omega_0^{(\mu)}/\Omega_0$ are constant in time for each ion, we can design pulse sequences that perform almost the same operation on each instance.

A. Composite rotations

The pulse $P_{ie}(\phi,\theta)$ is driven by a Hamiltonian $\tilde{H}_1=\frac{1}{2}\mathbf{\Omega}\cdot\boldsymbol{\sigma}^{(ie)}$ with $\boldsymbol{\sigma}^{(ie)}$ signifying the Pauli matrices in the $\{|i\rangle,|e\rangle\}$ basis and $\mathbf{\Omega}=\Omega_0\hat{\boldsymbol{n}}_{\phi}$, where $\hat{\boldsymbol{n}}_{\phi}$ is a unit vector in the x-y plane with azimuthal angle ϕ.

To apply the pulse $P_{ie}(\phi,\theta)$, we engage the field for a period θ/Ω_0, so that an ideal reference ion ξ with $\delta^{(\xi)}=0$ and $\Omega_0^{(\xi)}=\Omega_0$ will be rotated by an angle θ around $\hat{\boldsymbol{n}}_{\phi}$ as desired. In general, however, the ions will react differently to the pulse due to their different detunings and coupling strengths.

The problem of taking all the ions through the same evolution when they react differently to the pulses has been studied in great detail in the magnetic resonance community [6]. Inspired by the discussion in Ref. [7], we have used the BB1 pulse sequence to replace a single pulse $P(0,\theta)$ with the following sequence of pulses:

$$P_{\text{BB1}}(0,\theta) = P(0,\theta/2)P(\phi_c,\pi)P(3\phi_c,2\pi)P(\phi_c,\pi)P(0,\theta/2). \tag{3}$$

For our reference ion ξ, the unitary evolution ${}^{\xi}P_{\text{BB1}}(\phi,\theta)$ caused by the $P_{\text{BB1}}(\phi,\theta)$ composite pulse is seen to be exactly identical to the evolution caused by $P(\phi,\theta)$. The use of five pulses for this simple task is justified, however, if we instead consider evolution ${}^{\mu}P_{\text{BB1}}$ of a general ion subject to the Hamiltonian

$$H_1^{(\mu)}=-\delta^{(\mu)}|e\rangle\langle e|+\tfrac{1}{2}\Omega_0^{(\mu)}\hat{\boldsymbol{n}}_{\phi}\cdot\boldsymbol{\sigma}^{(ie)}. \tag{4}$$

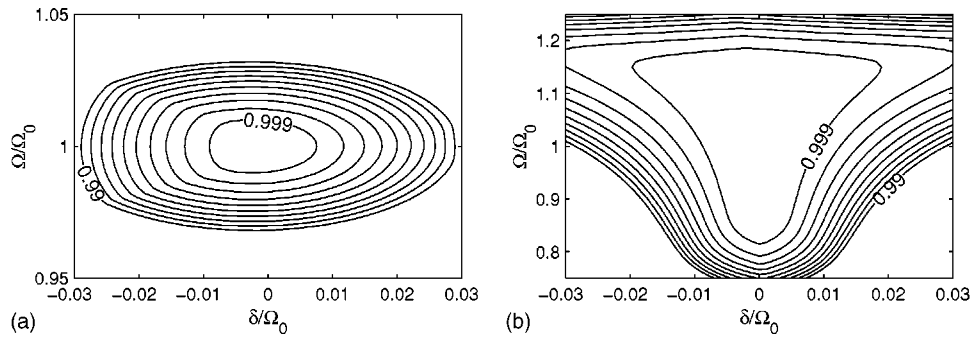

FIG. 1. Calculated worst case fidelities (6) of two implementations of the controlled phase shift: (a) the simple implementation (2) and (b) the $P_{\rm CPS}$ pulse sequence (5). The fidelity is plotted as a function of $\delta^{(1)}=\delta^{(2)}$ and $\Omega_0^{(1)}=\Omega_0^{(2)}$, both relative to Ω_0, and with $g_{12}=100\,\Omega_0$. Note the difference between the Ω-axis limits of the two plots. It is clear from the plots that $P_{\rm CPS}$ achieves a high fidelity over a much larger parameter space. In particular, $P_{\rm CPS}$ is much less sensitive to variations in Ω, while the sensitivity to variations in δ does not seem to be significantly improved.

In this case, we find that with the optimal value $\phi_c=\pm\cos^{-1}(-\theta/4\pi)$, ${}^{\mu}P_{\rm BB1}(\phi,\theta)$ is almost constant over a large range of values of $\delta^{(\mu)}/\Omega_0$ and $\Omega^{(\mu)}/\Omega_0$, while $\mu P(\phi,\theta)$ changes quite rapidly.

B. Robust gate operation

For the two-level Rabi problem, there is a global phase factor depending on the detuning which plays no observable role. In our three-level system, however, this phase will lead to a dephasing between the qubit level $|i\rangle$ coupled to $|e\rangle$ and the other qubit level. To compensate this, we must symmetrize the desired pulse sequence in a suitable way, to allow both levels to pick up the same, unknown, phase contributions.

In the case of the controlled phase shift (2), we have arrived at the following symmetrized version:

$$\begin{aligned} P_{\rm CPS}^{(i,j)} = {} & P_{1e}^{(i)}(\pi,\pi) \\ & \times P_{0e}^{(j)}(\pi,\pi)P_{0e}^{(j)}(0,\pi)P_{1e}^{(j)}(\pi,\pi)P_{1e}^{(j)}(0,\pi) \\ & \times P_{1e}^{(i)}(0,\pi)P_{0e}^{(i)}(\pi,\pi) \\ & \times P_{0e}^{(j)}(\pi,\pi)P_{0e}^{(j)}(0,\pi)P_{1e}^{(j)}(0,\pi)P_{1e}^{(j)}(0,\pi) \\ & \times P_{0e}^{(i)}(0,\pi). \end{aligned} \tag{5}$$

For the reference ion ξ, the $P_{\rm CPS}^{(i,j)}$ pulse sequence is seen to be equivalent to $P_{0e}^{(i)}(\pi,\pi)\ P_{1e}^{(j)}(0,2\pi)\ P_{0e}^{(i)}(0,\pi)$, which is exactly the basic controlled phase-shift operation (2), but we expect it to perform better for a general ion.

Implementing all the pulses of $P_{\rm CPS}^{(i,j)}$ by composite BB1 pulses (3), we do indeed obtain a very robust implementation of the controlled phase shift as illustrated in Fig. 1. To assess the gate performance, we have compared the effect of the gate to the desired gate operation $U_{\rm CPS}$ in terms of the worst case fidelity, defined as the minimal overlap between the actual outcome of the pulse sequence and the desired outcome of the gate operation:

$$\mathcal{F}(U_{\rm CPS},{}^{\mu}P)=\min_{|\psi\rangle\in\mathcal{H}}|\langle\psi|U_{\rm CPS}^{\dagger}\,{}^{\mu}P|\psi\rangle|^2. \tag{6}$$

Since we know that the starting point of the gate operation will be a superposition of the ground hyperfine states, we have not minimized the expression (6) over the full Hilbert space, but rather restricted $\mathcal{H}$ to the qubit space. Note that this modification ensures that any population in the excited state after the gate operation is counted as a loss of fidelity as it should be. The computation of the fidelity is discussed in more detail in the Appendix.

As we see from Fig. 1, the pulse sequence $P_{\rm CPS}$ obtains high fidelities over a much larger parameter space than the simple gate operation described by Eq. (2). This is highly desirable, as the minimal fidelity among the included instances determines the scale up needed to perform the error correction [8]. Not too surprisingly, the sensitivity to variations in Ω is improved the most, as this is the type of error best dealt with by the BB1 pulse sequence. For realistic parameters of the REQC system, a reduced sensitivity to δ variation would be more useful; whether this can be achieved by means of composite pulses is a point of further study.

IV. THE BUS ARCHITECTURE

One of the exciting features of REQC is that the size and coupling topology of the quantum computer is not defined by the crystal, but rather chosen in an initialization stage at each start up of the system. The choice of architecture determines the number of instances available in a given crystal, and thus ultimately the scaling properties of the system. The fully interconnected "cluster" architecture suggested in the original REQC proposal, of course, has the minimal topological distance between qubits. On the other hand, a star topology with one central qubit coupled to the $n-1$ remaining qubits, as illustrated in Fig. 2, would reduce the number of required couplings from $n(n-1)/2$ to $n-1$, thus increasing the number of available instances in a given crystal, while still maintaining a topological distance of only 2.

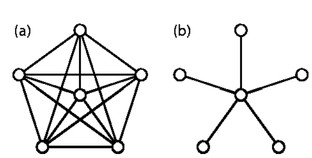

FIG. 2. Two possible coupling topologies for REQC systems: (a) cluster topology and (b) star topology.

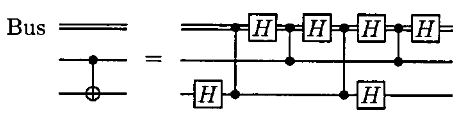

FIG. 3. A bus-mediated controlled-NOT gate based on controlled phase gates (vertical lines) and Hadamard operations H.

Since the outer qubits in the star topology are not directly coupled, two-qubit gates between those must be mediated by the central qubit acting as a *bus*. To be specific, a bus-mediated controlled NOT could be constructed from controlled phase gates and Hadamard operations as illustrated in Fig. 3.

In addition to the better scaling properties of the bus architecture, its main advantage is that the bus qubit is a participant of all multiqubit gates. This fact can be used to ease or improve the implementation of such gates. As an example, four times as many pulses are needed on channel j as on channel i with the proposed implementation of the controlled phase shift (5). If channel j is chosen as the bus channel, a dedicated laser system can speed up the application of these pulses as compared to a tunable laser system able to address any channel.

V. PREPARATION AND DETECTION OF MAXIMALLY ENTANGLED STATES

To demonstrate the viability of the REQC concept, and, in particular, the bus architecture, we propose to perform an experimental preparation and detection of a maximally entangled state.

We will use an REQC system with the star topology: one central qubit coupled to $n-1$ outer qubits. Starting with all n qubits in their $|0\rangle$ state, we apply a composite pulse Hadamard operation to the central qubit followed by controlled-not operations on all the outer qubits controlled by the central qubit, thus transferring the system to the maximally entangled state

$$|\Psi_0\rangle=(1/\sqrt{2})(|0\rangle^n+|1\rangle^n), \tag{7}$$

which corresponds to a superposition of the total pseudospin pointing straight up and straight down.

The following algorithm for detecting a population of the state $|\Psi_0\rangle$ is very similar to the method used by the group of Wineland to detect a maximally entangled state of four ions in a linear Paul trap [9]: By rotating the state $|\Psi_0\rangle$ through an angle ϕ around the $\hat{\boldsymbol{z}}$ axis, we accumulate different phases on the pseudospin components: $|\Psi_1\rangle=(1/\sqrt{2})(|0\rangle^n+e^{-i\phi n}|1\rangle^n)$. An additional rotation by $\pi/2$ around the y axis now yields a state $|\Psi_2\rangle$ with an expected parity $P=\Pi_i(\sigma_z)_i$ given by

$$\langle\Psi_2|P|\Psi_2\rangle=\cos(n\phi),$$

the detection of the $n\phi$ dependency thus signifying that the maximally entangled state has been populated [10].

In a single-instance quantum computing system, such as the ion trap setup used in Ref. [9], we could measure the expectation value of the parity as a statistical average over many repetitions of the procedure described above: after each run, we could simply measure the state of each qubit, and subsequently compute the parity. Since measurements in the REQC system yield an ensemble average, this approach would not be applicable here; we cannot find the expectation value of the parity from the ensemble averages of the single qubit parities $\langle(\sigma_z)_i\rangle$, which are 0 as the inspection shows.

Instead we let the bus qubit acquire the parity unitarily: by sequentially applying controlled-NOT operations from each outer qubit to the central qubit, we make the central qubit end up in the $|1\rangle$ state in the case of odd parity and in the $|0\rangle$ state in the case of even parity. After this, the ensemble average of the bus qubit population yields the expectation value of the parity.

As this section illustrates, readout from an ensemble quantum computer is conceptually somewhat more complicated than readout from a single quantum computer. It is worth noting, however, that unlike many other ensemble quantum computing proposals, REQC instances all start in the same pure state; if we successfully employ error correction during a computation, all instances will end up in the same pure state, allowing us to read out the ensemble averages with high signal to noise ratio. Perhaps surprisingly, the readout can almost always be performed by tricks similar to those employed to detect the maximally entangled state; ensemble quantum computing is almost as powerful as general quantum computing. In particular, all problems, which may be expressed in terms of the hidden subgroup problem (such as Shor's factoring algorithm), can be solved using an ensemble quantum computer [11].

VI. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown that, in the absence of decay and decoherence, it is possible to implement robust high-fidelity gates for the REQC system. Specifically, the phase compensated controlled phase gate based on composite pulses (5) achieves worst case gate fidelities above 0.999, even with the coupling strength varying up to 10% between instances and channel widths of several percent of the Rabi frequency of the field used to manipulate the system. Furthermore, we have pointed out that using a bus based architecture will simplify implementation by allowing the use of an asymmetric laser setup.

The number of instances of a bus based REQC system scales as p^n where n is the number of qubits per instance and p is the probability of a random ion being coupled to a member of a given channel. In the regime currently being investigated experimentally, p is several orders of magnitude less than 1. The value of p is affected by g_t and channel width, which is why we have to use robust gates rather than narrow channels and high threshold coupling strengths. Higher values of p could be obtained by increasing the ion density, which would, however, cause a decrease in coherence times. By using structured doping techniques, it might be possible to obtain a higher effective p without this adverse effect. Another approach to obtaining higher effective p would be to use multiple channels for each qubit by guaranteeing each instance to have exactly one member ion from a group of channels assigned to each qubit.

The instance identification protocol described in Sec. II could be made much more efficient; since the system starts in a pure state (all ions in the channels in their $|0\rangle$ state), and also ends in a pure state (all instance members in their $|0\rangle$ state, and all the other ions from the initial channel populations in their $|\mathrm{aux}\rangle$ state), the selection could theoretically be performed unitarily.

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APPENDIX: FIDELITY OF UNITARY OPERATIONS

We wish to compare unitary operators U and U_0, by determining how closely $U_0^\dagger U$ resembles the identity on the Hilbert space $\mathcal{H}$. This can be expressed in terms of the worst case fidelity:

$$\mathcal{F}(U_0,U)=\min_{|\psi\rangle\in\mathcal{H}}|\langle\psi|U_0^\dagger U|\psi\rangle|^2. \tag{A1}$$

The fidelity can be computed as follows: $U_0^\dagger U$ is unitary and can consequently be formally diagonalized with eigenvalues $e^{i\phi_j}$, $j=1,\ldots,n$ so that $0\leqslant\phi_1\leqslant\cdots\leqslant\phi_n\leqslant 2\pi$. Introducing the maximal eigenvalue phase distance $\Delta\phi_{\max}=\max(\{\phi_j-\phi_{j-1}\}_{j=2,\ldots,n}\cup\{2\pi+\phi_1-\phi_n\})$, the fidelity over $\mathcal{H}$ is given as

$$\mathcal{F}(U_0,U)=\begin{cases}\cos^2(\Delta\phi_{\max}/2) & \text{if } \Delta\phi_{\max}\geqslant\pi,\\ 0 & \text{otherwise.}\end{cases} \tag{A2}$$

To see this, we expand the state vector $|\psi\rangle$ on the eigenbasis $\{|j\rangle\}$ of $U_0^\dagger U$: $|\psi\rangle=\Sigma_j c_j|j\rangle$. The fidelity then takes the form

$$\mathcal{F}(U_0,U)=\min_{p_j}\left|\sum_j p_j e^{i\phi_j}\right|^2, \tag{A3}$$

with the minimum taken over all non-negative $p_j=|c_j|^2$, so that $\Sigma_j p_j=1$.

Equation (A3) allows us to interpret the fidelity geometrically in the complex plane, as the set of points $\{\Sigma_j p_j\exp(i\phi_j)\}$ forms a convex polygon with vertices in the eigenvalues $\{e^{i\phi_j}\}$ on the unit circle. The fidelity corresponds to the square of the minimal distance from 0 to this polygon. If the polygon is constrained to one half plane, this will be $|e^{i\phi}+e^{i(\phi+\Delta\phi_{\max})}|^2/4=\cos^2(\Delta\phi_{\max}/2)$. If the polygon is not restricted to one half plane, it will cover the origin, and the fidelity will be 0.

Note that this method relies on the minimization being performed on the whole Hilbert space. If this is not the case, the method is not applicable, and in Sec. III B where the minimization is carried out over a subspace of the full Hilbert space, we have resorted to a numerical search.

[1] M.D. Lukin and P.R. Hemmer, Phys. Rev. Lett. **84**, 2818 (2000).

[2] J.J. Longdell and M.J. Sellars, e-print quant-ph/0208182.

[3] N. Ohlsson, R.K. Mohan, and S. Kröll, Opt. Commun. **201**, 71 (2002).

[4] M. Nilsson, L. Rippe, N. Ohlsson, T. Christiansson, and S. Kröll, Phys. Scr. **T102**, 178 (2002).

[5] D. Jaksch, J.I. Cirac, P. Zoller, S.L. Rolston, R. Côté, and M.D. Lukin, Phys. Rev. Lett. **85**, 2208 (2000).

[6] M.H. Levitt, Prog. Nucl. Magn. Reson. Spectrosc. **18**, 61 (1986).

[7] H.K. Cummins, G. Llewellyn, and J.A. Jones Phys. Rev. A **67**, 042308 (2003).

[8] A.M. Steane, e-print quant-ph/0207119.

[9] C.A. Sackett *et al.*, Nature (London) **404**, 256 (2000).

[10] J.J. Bollinger, W.M. Itano, D.J. Wineland, and D.J. Heinzen, Phys. Rev. A **54**, R4649 (1996).

[11] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).