Mean mean values

The field inside a random distribution of parallel dipoles

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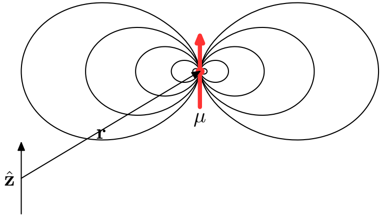
May 29, 2004

The field from a dipole parallel to the z-axis is well known:



$$E_z = \frac{\mu}{4\pi\varepsilon_0}\frac{1}{r^3}\left((\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\mu}}) - 3(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}})(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\mu}})\right)$$

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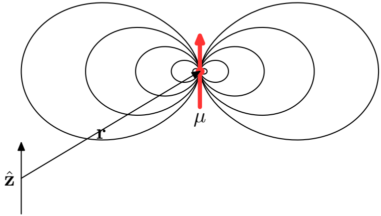

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\begin{aligned}
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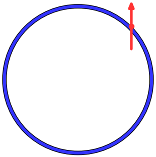
&= \frac{\mu}{4\pi\varepsilon_0}\frac{1}{r^3}(1 - 3\cos^2(\theta))
\end{aligned}
$$

We will consider the expected field due to randomly placed dipoles parallel to the z-axis.



Single dipole on a spherical shell:

$$\langle E_z \rangle \propto \int (1 - 3\cos^2(\theta)) d\Omega = 0$$

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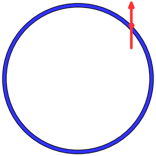

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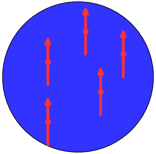
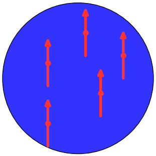


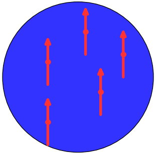
In a sphere with dipole density ρ:

$$\langle E_z \rangle \stackrel{?}{=} 0$$



No, $\langle E_z \rangle \neq 0$ in a sphere with dipoles:

- $\langle E_z \rangle$ is not well defined.



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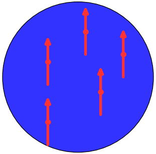

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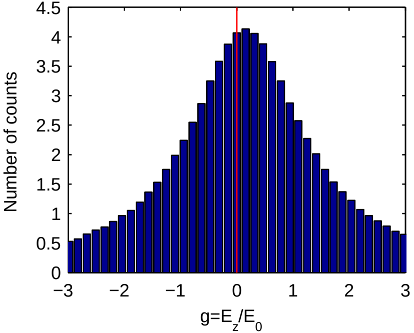

$$E_0 = \frac{\mu}{4\pi\varepsilon_0}\frac{1}{r_0^3}$$

$$\frac{4\pi}{3}r_0^3 = \frac{1}{\rho}$$

Talk outline

Field of a random distribution of dipoles

Modeling spatial correlations

Experimental observations

Conclusions

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$$g = \left(\frac{r}{r_0}\right)^{-3} \left(1 - \cos^2\theta\right)$$

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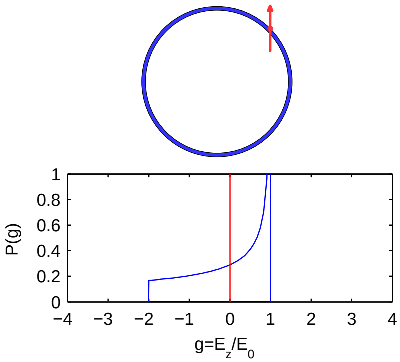

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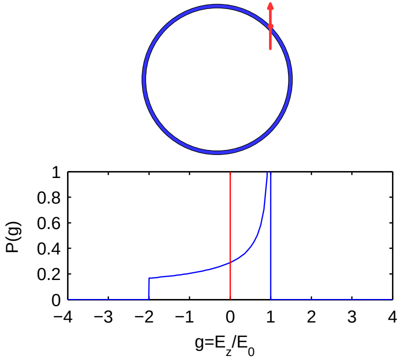
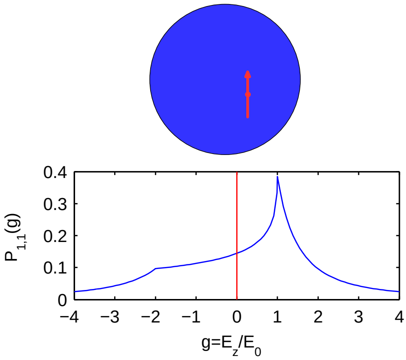
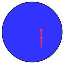


$\langle E_z \rangle = 0$

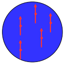


$\langle E_z \rangle$: ill-defined

The distribution of the field due to N dipoles can be expressed as a folding integral:



$P_{1,N}(g)$: distribution corresponding to a single dipole in a sphere of radius $\sqrt[3]{N}r_0$



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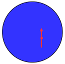

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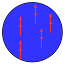

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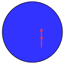

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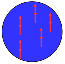

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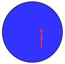

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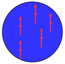

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Taking the limit of $N \to \infty$

$$\lim_{N\to\infty} \log \tilde{\mathrm{P}}_N(k) = -\Gamma|k| - ig_c k$$

From $\log \tilde{\mathrm{P}}_\infty(k) = -\Gamma|k| - ig_c k$ follows that

$$P_\infty(g) = \frac{1}{\pi}\frac{\Gamma}{\Gamma^2 + (g - g_c)^2}$$

$$g_c = \frac{2}{9}\left(3 + \sqrt{3}\log\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \approx 0.1598$$

$$\Gamma = \frac{2\pi}{\sqrt{27}} \approx 1.2092$$

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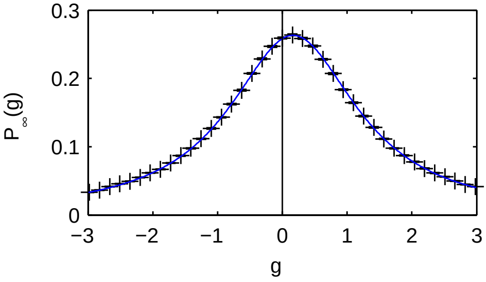
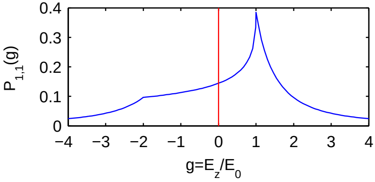


The figure is based on numerical simulations of $1.000.000$ realizations of a system with 50.000 dipoles.

Understanding g_c

Although $\langle g \rangle$ is undefined, g_c can be obtained as the expectation value of g if $P_{1,1}(g)$ is truncated symmetrically around 0:



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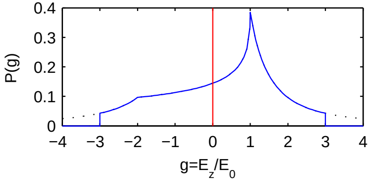
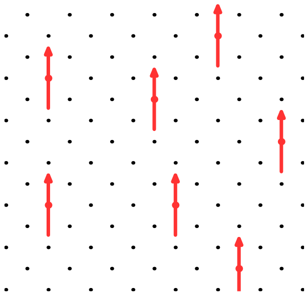


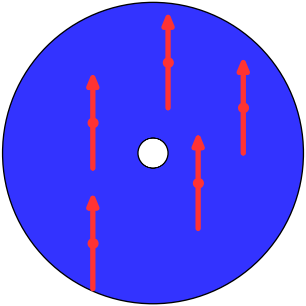
$\langle g \rangle = g_c$

Spatial correlations

Due to finite size effects, dipoles will not be placed independently.



To model spatial correlations we introduce an *excluded volume* of radius $\varepsilon^{1/3} r_0$ (so that ε is the number of dipoles that would have fitted inside the excluded volume).



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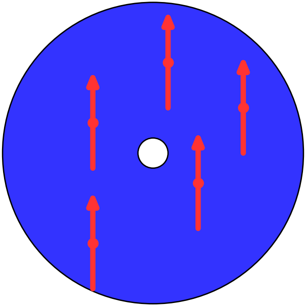
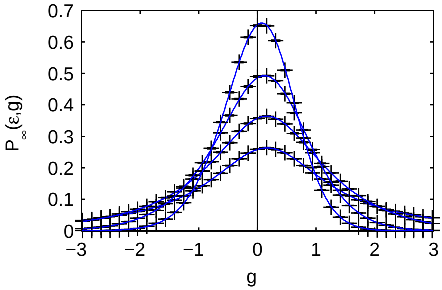


$$\langle E_z \rangle \propto \int_{\varepsilon^{1/3} r_0}^{N^{1/3} r_0} r^{-3}\, r^2 dr \int (1 - 3\cos^2\theta) d\Omega = 0$$

When an excluded volume is included, the analytical form of the probability distribution is found to be

$$\log \tilde{\mathrm{P}}_{\infty}(\varepsilon, k) = \log \tilde{\mathrm{P}}_{\infty}(k) - \varepsilon \left(\tilde{\mathrm{P}}_{1,1}\left(\frac{k}{\varepsilon}\right) - 1 \right).$$



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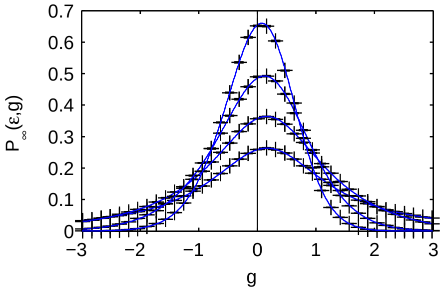

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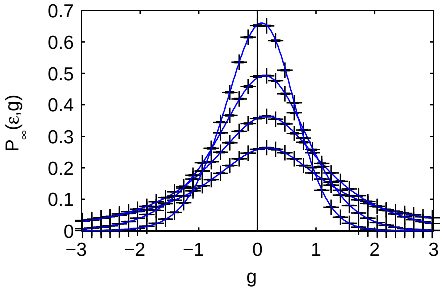

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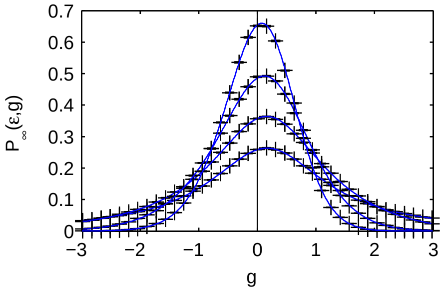
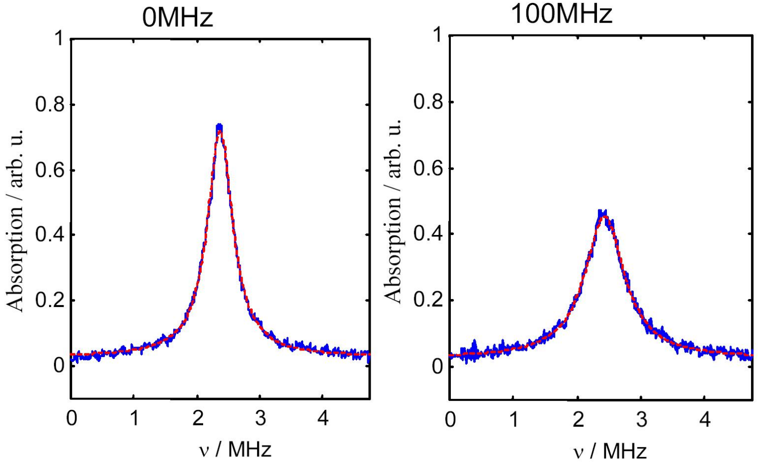


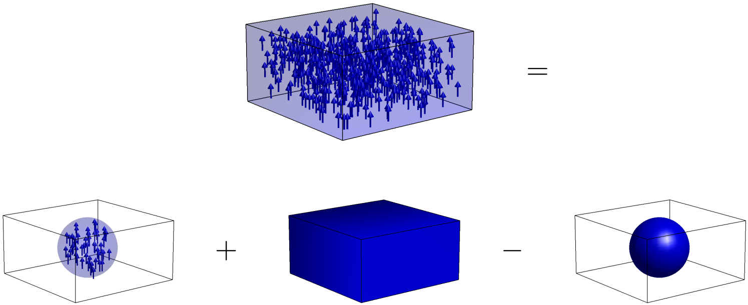
- Shifted Lorentzian for $\varepsilon \to 0$
- Gaussian with $\sigma^2 \propto \varepsilon^{-1}$ for $\varepsilon \to \infty$
- $\langle g \rangle = 0$ for all $\varepsilon \neq 0$

Lorentzian broadening has been observed for a mix of anti-parallel dipoles where no shift is expected:



[Nilsson *et.al.*, Phys. Scrip. **T102**, 178 (2002)]

Measuring the shift is harder due to the macroscopic field associated with a non-vanishing polarization.



$$\boldsymbol{E}^{\text{micro}} = \boldsymbol{E}^{\text{micro}}_{\text{near}} + (\boldsymbol{E}^{\text{macro}} - \boldsymbol{E}^{\text{macro}}_{\text{near}})$$

Two examples: box-shaped and spherical samples

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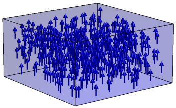

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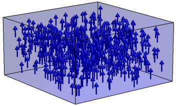

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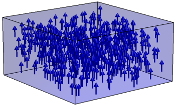

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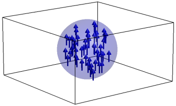

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Conclusions

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- Including spatial correlations, we find a family of distributions, all with vanishing mean, approaching the shifted Lorentzian uniformly.
- Although the Lorentzian broadening is well known and has been observed, the shift effect apparently requires a dedicated experiment.

Acknowledgments

Francois Bardou, Université Louis Pasteur, has offered many helpful references for this work.

This work was funded by the European Union IST-FET programme ESQUIRE.

$$g = \frac{F_z}{F_0} = \left(\frac{r}{r_0}\right)^{-3} d(\hat{\boldsymbol{r}}, \hat{\boldsymbol{n}}),$$

$$P_{1,N}(g) = \left\langle \int_0^{N^{1/3} r_0} \delta \left(g - \left(\frac{r}{r_0}\right)^{-3} d(\hat{\boldsymbol{r}}, \hat{\boldsymbol{n}}) \right) \frac{3r^2 dr}{N r_0^3} \right\rangle$$

$$P_{1,N}(g) = \frac{1}{N g^2} D(Ng),$$

where $D(g)$ is a geometrical factor which depends only on the distribution of d:

$$D(g) = \left\langle |d(\hat{\boldsymbol{r}}, \hat{\boldsymbol{n}})| \int_0^1 \delta(u - \frac{d(\hat{\boldsymbol{r}}, \hat{\boldsymbol{n}})}{g}) du \right\rangle.$$

$$D^{(p)}(g) = \frac{1}{3\sqrt{3}} \begin{cases} 2 - (2+g)\sqrt{1-g} & \text{if } -2 < g < 1, \\ 2 & \text{otherwise.} \end{cases}$$